

The Cellular Automata $*0*0*000$ Have No Nontrivial Conserved Functions

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Abstract

This paper provides a proof for a statement made by the authors in an earlier paper [1] that none of the 1D, 2-state, 3-neighborhood cellular automata of the form $*0*0*000$ (where each $*$ is either 0 or 1) have any nontrivial conserved functions.

Definitions

A famous class of cellular automata is those that are one-dimensional, have only two states per cell, and each cell is updated on each cycle as a function of itself and its two neighbors. There are 256 such automata, and the standard name for each one is found by converting an 8-bit binary number to decimal. Each of the 8 bits gives the rule for the new state of a cell as a function of its current state and the state of its two neighbors. For example, CA 168 is:

$$168_{10} = 10101000_2 \quad \Rightarrow \quad \begin{array}{l} 111 \rightarrow 1 \\ 110 \rightarrow 0 \\ 101 \rightarrow 1 \\ 100 \rightarrow 0 \\ 011 \rightarrow 1 \\ 010 \rightarrow 0 \\ 001 \rightarrow 0 \\ 000 \rightarrow 0 \end{array}$$

There are 8 possible CAs with the binary pattern $*0*0*000$, where each $*$ can be 0 or 1. It will be shown that these 8 CAs have a particular property in common.

An *energy function of order n* is a function $f: \{0,1\}^n \rightarrow \mathbb{Z}$ that assigns an integer-valued energy to any possible state of a CA world of any size (assuming that the world wraps around, so the left and right ends are defined to be neighbors). It works by applying f to every possible window of size n , and summing the results.

An energy function is *conserved*, if the energy assigned to a world is unchanged by running the CA one step. The energy function is *trivial* if it assigns the same energy to all worlds of each size, for all possible sizes. A cellular automaton is said to have a *nontrivial conserved function* if there is a function f of order n that is not trivial and is conserved for worlds of all possible sizes.

Theorem and proof

We now prove the following theorem:

Theorem There are no nontrivial conserved functions for the 8 cellular automata of the form $*0*0*000$, where $*$ is 0 or 1.

Proof

First, we will prove this for the cellular automaton 168 shown above, then show the same for the other 7 CAs of the given form.

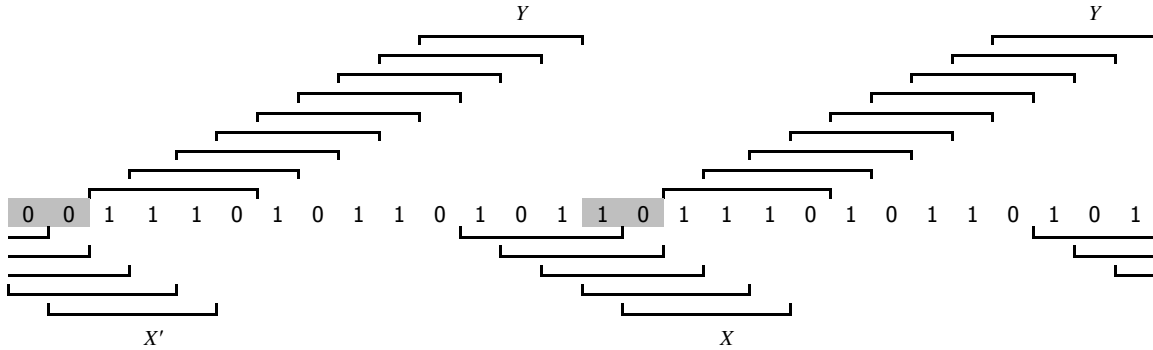
CA 168 allows a cell to remain a 1 on the next time step if and only if it and its two neighbors are currently 111, 101 or 011. The three patterns 000, 001, and 100 all go to 0, so any pair of adjacent cells 00 will remain 00 after any number of iterations. Both 110 and 010 go to 0, so whenever there is a 1 to the left of the 00 it will also become a 0 and stay 0 forever. Therefore, if the pair 00 occurs anywhere in the world, it will eventually grow to the left until it fills the entire world with zeros. For example, the following shows several iterations of CA 168 on a world of size 25 for a particular starting state:

```
1111101111110011111101011
1111011111100011111010111
1110111111000011110101111
1101111110000011101011111
1011111100000011010111111
0111111000000010101111111
1111110000000001011111110
111110000000000111111101
11110000000000011111011
111000000000000111110111
110000000000000111101111
100000000000000111011111
000000000000000110111111
000000000000000101111110
00000000000000011111100
00000000000000011111000
00000000000000011110000
00000000000000011100000
00000000000000011000000
00000000000000010000000
00000000000000011000000
00000000000000010000000
0000000000000000000000000
0000000000000000000000000
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In order for the function f to be conserved, it must assign the same energy to every state that contains at least one 00 pair, because all such states eventually evolve into the same state (the all-0 state), and each state must have the same energy as its successor (if energy is conserved).

Now consider an arbitrary conserved function f of order n , and an arbitrary state S of length at least $n+1$. Let S' be the state S with the first two cells changed to 00. Let string concatenation be represented by adjacency, so SS' is string S concatenated with S' , and S^m is m copies of S concatenated together. Note that if S is of size at least $n+1$, then a single window of length n cannot simultaneously include part of the first two bits of both S' strings in $S'S'$.

Let X' be the sum of f applied to every window in S' (including those that wrap around) that contains at least one of the first two bits of S' (which are 00). Let X be the sum of f applied to windows in those same positions for the string S . Let Y be the sum of f applied to all the other windows in S . For example, if $S=10111010110101$ and $n=4$ then for $S'S$, f is applied to all the bracketed windows, and the sums are:



This diagram shows S' and S with their first two cells in gray, the windows for X' (bottom middle), X (bottom left and right), one Y (above left), and the other Y (above right). The total energy for $S'S$ is therefore $2Y+X+X'$. Let E and E' be the energies of the following four states:

$$\begin{aligned}
 E &= \text{energy}(SS) = 2Y + 2X \\
 E' &= \text{energy}(S'S') = 2Y + 2X' \\
 E' &= \text{energy}(S'S) = 2Y + X + X' \\
 E' &= \text{energy}(SS') = 2Y + X + X'
 \end{aligned}$$

The energy must be equal for the last three equations, because f is conserved, and each of those states contains at least one 00 pair. Add the first two equations, subtract the other two, and simplify to get:

$$\begin{aligned}
 (E) + (E') - (E') - (E') &= (2Y + 2X) + (2Y + 2X') - (2Y + X + X') - (2Y + X + X') \\
 E - E' &= 0 \\
 E &= E'
 \end{aligned}$$

Therefore, all world states of the form SS (for S of size at least $n+1$) have the same energy, whether they contain the string 00 or not.

Now consider a world of any positive size, and consider two states for that world, T and U . Because the worlds wrap around, the energy for T will be exactly $1/(2(n+1))$ times the energy for a larger world with state $T^{n+1} T^{n+1}$. Similarly, the energy for U will be exactly $1/(2(n+1))$ times the energy for the state $U^{n+1} U^{n+1}$. But those two larger states must have equal energy, because they are both of the form SS for an S of length at least $n+1$. Therefore T and U must have the same energy.

Thus, for any world of any positive size, the conserved energy function f yields the same energy on all states for that world. So every conserved f is trivial. So CA 168 has no nontrivial conserved functions.

Finally, consider changing CA 168 (which is 10101000) by changing one or more of the 1 bits in its name to 0. The above argument relied only on the property that a 00 pattern could never disappear, and would always grow until it filled the entire world. Clearly, this property is preserved if the CA is changed but continues to produce 0 bits in all the cases that it did in CA168, plus some additional cases. Therefore, there are no nontrivial conserved functions for any CA whose name is formed by taking 10101000 and setting zero or more of the 1 bits to 0. Thus any cellular automaton of the form *0*0*000 has no nontrivial conserved functions. □

Equivalence Classes

As mentioned in the previous paper [1], two CAs are in the same equivalence class if every history for one can be converted to the other by inverting all the bits (1 becomes 0 and vice versa), or reversing the image left/right, or both. If a CA has no nontrivial conserved functions, then neither will the other members of its equivalence class. For completeness, we now list the 8 CAs analyzed here, plus all others in their equivalence classes, for a total of 24 CAs that have no nontrivial conserved functions.

x0x0x000	Reverse	Invert	Both
0		255	
8	64	239	253
32		251	
40	96	235	249
128		254	
136	192	238	252
160		250	
168	224	234	248

Reference

[1] Baird, Leemon C. III & Fagin, Barry (2007) "Conservation functions for 1-D automata: Efficient algorithms, new results, and a partial taxonomy", *Journal of Cellular Automata*.